Proving an Equivalence Between the Hadwiger-Nelson Problem and a Polygon Generalization

Joint work with Sean Fiscus and Hongyi Zhang

The Hadwiger-Nelson Problem

The Hadwiger-Nelson problem has been a major unanswered question in Geometric Combinatorics since it was first asked in the 1950s. Its popularization by several leading graph theorists and geometers, most famously Paul Erdős, led to the development of the field of Euclidean Ramsey Theory [1]. Problems in Euclidean Ramsey Theory, including this one, typically lie at the intersection of geometry, graph theory, measure theory, and the study of set theory axioms, providing a testing ground for a diverse range of techniques, ideas, and methods developed in other fields. The Hadwiger-Nelson problem has also propelled the study of unit distance graphs, whose applications range from configuring satellites [2] to image processing [3].

The problem asks for the minimum number of colors required so that that every point in $\mathbb{R}^2$, the Cartesian plane, can be assigned a color such that that no two points at distance 1 apart are the same color. To be concise, I will henceforth refer to this as the minimum number of colors needed to monochromatically forbid distance 1. The best known upper bound on the number of colors needed is 7, which can be seen by tiling the plane with hexagons of side length $2/5$ as shown in Figure 1, and noting that indeed no two points that have the same color in the figure are a distance 1 apart [1]. The best known lower bound stood at 4 for over 60 years, until computer scientist Aubrey de Grey proved that at least 5 colors are required to forbid distance 1 [4]. His proof used a computer to generate a graph with 1581 vertices that could be embedded into the plane so that vertices connected by an edge are exactly a distance 1 apart, and then verified that the chromatic number of this graph is at least 5.

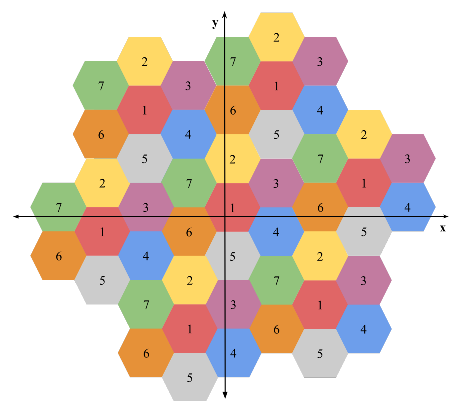

Figure 1: $\mathbb{R}^2$ colored with Hadiwger Tiles. Colors are denoted by numbers 1 through 7.

Generalizations and Related Problems

Several generalizations of the Hadwiger-Nelson problem have become a major combinatorial research area. One of the most famous generalizations asks for the following: Given a polygon, by which I mean the finite set of points $T = \{p_1, ..., p_n\} \subset \mathbb{R}^2$ that form its vertices, what is the minimum number of colors needed to forbid monochromatic copies of T? More explicitly, we are asking for the minimum number of colors needed to color the plane so that no polygon congruent to T is monochromatic. The Hadwiger-Nelson problem as stated above is precisely this problem when T consists of two points a distance 1 apart.

Much work has been done to find bounds on the number of colors needed to forbid various triangles. For example, in [5], Paul Erdős and several collaborators proved that if T is an equilateral triangle of side length 1, then only 2 colors are sufficient to forbid monochromatic copies of T. This can be done by coloring $\mathbb{R}^2$ with infinite vertical strips of width $\sqrt{3}/2$, which is equal to the height of the triangle, as shown in Figure 2. A moment of inspection shows that the vertices of any unit-side-length equilateral triangle are never all red or all blue.

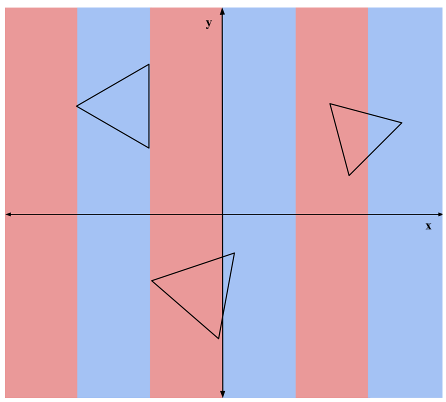

Figure 2: $\mathbb{R}^2$ colored with vertical strips. Three unit length equilateral triangles are pictured, each of which has a vertex lying on both a blue and a red point.

They also conjecture that if T is a triangle that is not an equilateral triangle, then at least 3 colors are needed. It is known that the number of colors varies significantly even with small adjustments to the polygon T. A final but important note is that we have thus far been coloring to forbid just a single polygon, but several authors including [5] and [6] have considered forbidding several polygons at once. Thus one can ask for the minimum number of colors required to forbid monochromatic polygons in a set $S = \{T_1, ..., T_k\}$, meaning that the vertices of no polygon should appear monochromatically.

In this project, we provide a new construction that proves an equivalence between the original Hadwiger-Nelson problem and a class of polygon generalizations. This shows for the first time that the two problems, previously thought to be unrelated, are in fact the same for a well-chosen choice of polygons.

Main Result

We began the project by defining a notion of equivalent coloring problems. It is often the case that two sets of polygons $S = \{T_1, ..., T_k\}$ and $P = \{L_1, ..., L_m\}$ need the same number of colors to be forbidden, but the particular colorings that achieve this are different. We will say that it is equivalent to forbid two sets S and P when not only the number of colors needed is equal, but also a coloring forbids S with this minimal number of colors if and only if it also forbids P. The main theorem we proved then shows that such an equivalence exists between the Hadwiger-Nelson Problem and the problem of forbidding some set S of triangles, or even n-gons for $n \geq 3$. The precise statement is given below.

Theorem 1. *For any $n \in \mathbb{N}$, $n \geq 3$, there exists a finite set S of n-gons such that forbidding all of them is equivalent to coloring the plane to monochromatically forbid distance 1.*

The proof relies on the notion of hypergraphs, which consist of a set called vertices, and a set of subsets called hyperedges. In the case where the hyperedges all have 2 elements, one recovers the standard definition of a graph. The proof then references a generalization of a theorem of Erdős and De Bruijn. It states that the vertices of a hypergraph can be colored so that no edge is monochromatic if and only if the vertices of every finite sub-hypergraph can be colored so that no edge is monochromatic. Using a recursive process, I define a new set of a geometrically-defined hypergraphs resulting from successive applications of this theorem. The termination of this construction yields a hypergraph, which after some modification, admits the same colorings that forbid monochromatic edges of cardinality 2, and monochromatic edges of cardinality n. The proof is non-constructive as the De Bruijn–Erdős theorem uses the axiom of choice.

Further Results and Generalizations
Theorem 1 holds more generally in $\mathbb{R}^d$ with the Euclidean norm, for any $d \geq 1$. After we proved the theorem, my collaborators and I started generalizing the result to other spaces. I proved a more general result that holds in any normed vector space over $\mathbb{R}$, albeit one must settle for a partial equivalence.

Theorem 2. *Let V be a normed $\mathbb{R}$-vector space, and let $n \in \mathbb{N}$, such that $n \geq 3$. Then there exists a finite set $\{T_1, ..., T_k\}$ of n-gons such that the number of colors needed to forbid $\{T_1, ..., T_k\}$ is equal to the number of colors needed to forbid distance 1.*

Conclusions and Further Research
The equivalences proved in this work suggest that there is a remarkable level of complexity in finding the number of colors to forbid $\{T_1, ..., T_k\}$. While intuitively it seems true that coloring to forbid a triangle is not as difficult as coloring to forbid a distance, we show that the two are, at least for some set of triangles, equally difficult.

This result shows that there exists some set $S = \{T_1, ..., T_k\}$ of n-gons with the desired property, which raises the question of how large this set S is. If for $n = 3$ one can find a set S of just one triangle, then a conjecture of Ronald Graham [7] can be disproved, which states that every triangle can be forbidden with 3 colors. With regards to Theorem 2, a conjecture of [5] states that S cannot have one element for sufficiently large $n \in \mathbb{N}$.

References

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